Basic concepts in mathematical optimization

Mathematical optimization (MO) (or mathematical programming) is a branch of applied mathematics, in which from many possible solutions the best one in a particular sense is sought. For example, out of many possible investments with set constraints, the most profitable is sought.

Mathematical optimization is widely used in economics, engineering, agriculture, logistics and other spheres.

To utilize MO the problem at hand has to be described using mathematical symbols, i.e. to create its corresponding mathematical model.

The model includes:

1. A function (or functions) of many variables, the optimal value of which is sought (max or min). This function is called objective.

2. A system of equations and inequalities which set the conditions and connect independent variables. These conditions are called problem constraints.

According to the type of function and constraints MO is subdivided into:

- Linear optimization (programming) – when the objective function and its constraints are linear;
- Non-linear optimization – when there is some kind of non-linearity in the objective function and/or in the constraints;
- Dynamic optimization – when one of the independent variables is time or the problem can be described as a successive multistage process.

A simple example of linear programming

Example 1. An investor is planning to buy shares from two companies: P_1 and P_2, having a part of no less than 12 000 euros in P_1 and no less than 30 000 euros in P_2. A single share from P_1 costs 1 500 euros, and a single share from P_2 – 2 000 euros. The total of the investment cannot exceed 100 000 euros.

a) compile the system of constraints for the model of the problem;

b) represent graphically the region of possible investments.

Solution: We mark with x – the number of shares from P_1, y – the number of shares from P_2.

a) The constraint for the investment in the first company is:

$$1500x \geq 12000.$$

Likewise for the second company:

$$2000y \geq 30000.$$

Constraint for the total price:

$$1500x + 2000y \leq 100000 .$$

Natural constraints for non-negativity of the number of shares *x* and *y*:

$$x \geq 0 , \ y \geq 0 .$$

This way we get the system of constraints for the model:

$$G : \left| \begin{array}{l} 1500x \geq 12000 \\ 2000y \geq 30000 \\ 1500x + 2000y \leq 100000 \\ x \geq 0 \\ y \geq 0 \end{array} \right. \tag{1}$$

After simplifying (1) assumes the form

$$G : \left| \begin{array}{l} 3x \geq 8 \\ y \geq 15 \\ 3x + 4y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{array} \right. \tag{2}$$

b) In order to represent graphically the region of investments *G*, we consider its corresponding system of equations:

$$\left| \begin{array}{ll} m_1 : & x = 8 \\ m_2 : & y = 15 \\ m_3 : & 3x + 4y = 200 \\ m_4 : & x = 0 \\ m_5 : & y = 0 \end{array} \right. \tag{3}$$

Each of the equations (3) can be considered as the equation of a line in the plane (*x*, *y*). Each line in turn divides the plane into two half-planes, in which exactly one the inequalities ≤ or ≥ is valid, for all points from respectively either of the half—planes or the other. In fig. 1 the half-planes, respectively (2) have been marked by hatches.

Noticeably the simplest way to construct a line in the plane is to find two arbitrary points from it. For example, for the line m_3 when $x = 0$ we calculate

$y = 50$, and when $y = 0$ we have $x = 66\frac{2}{3}$. Written down using point coordinates we have: $(0, 50)$, $\left(66\frac{2}{3}, 0\right)$. To find out in which half-plane the condition $3x + 4y \leq 200$ is satisfied, we check the sign of an arbitrary point from the plane. For example for point (0, 0) we have $3.0 + 4.0 = 0 \leq 200$. Consequently point (0, 0) satisfies the condition and we have to take the half-plane under line m_3.

The crossing points of the lines marked in fig 1 with *A*, *B*, *C* are solutions to the following three systems:

$$\text{т. } A = m_1 \times m_2 : \left| \begin{array}{l} x = 8 \\ y = 15 \end{array} \right. \qquad \Rightarrow \qquad A(8, 15);$$

$$\text{т. } B = m_2 \times m_3 : \left| \begin{array}{l} y = 15 \\ 3x + 4y = 200 \end{array} \right. \qquad \Rightarrow \qquad B(46\frac{2}{3}, 15);$$

$$\text{т. } C = m_1 \times m_3 : \left| \begin{array}{l} x = 8 \\ 3x + 4y = 200 \end{array} \right. \qquad \Rightarrow \qquad C(8, 44).$$

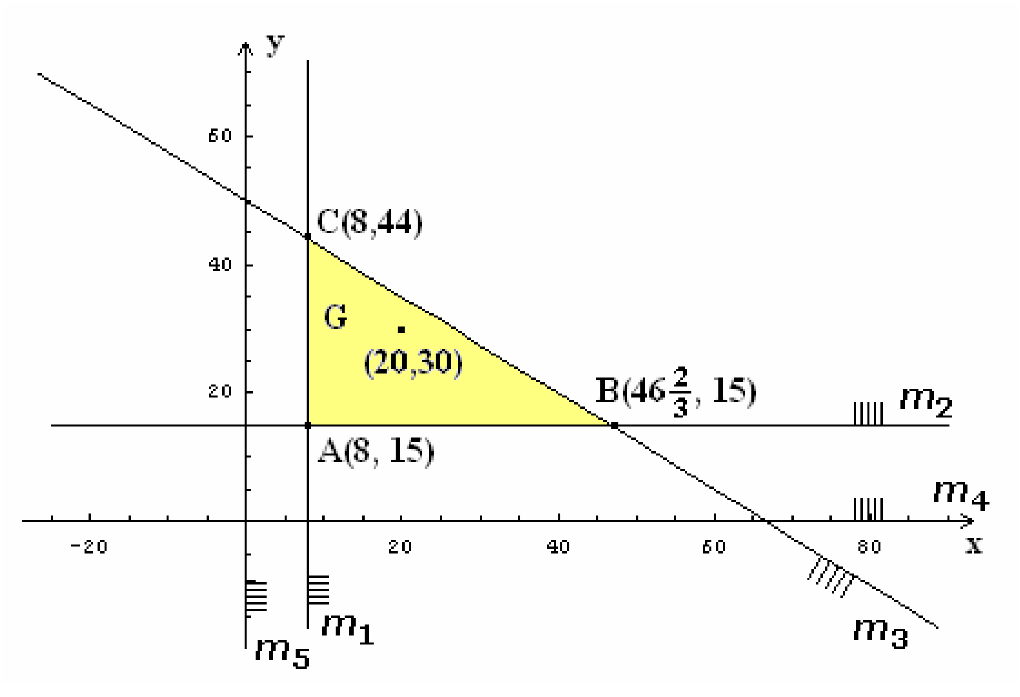

Fig. 1. Region of feasible solutions *G*.

Analysis: Each point from G is a possible investment because it satisfies the constraints of the problem.

For example point $(20,30) \in G$. Buying 20 shares from P_1 and 30 shares from P_2, the investor will pay a price of:

$1500.20 + 2000.30 = 30000 + 60000 = 90000$ euro.

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